Josephson vortex motion as a source for dissipation of superflow of e–h pairs in bilayers

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Abstract
It is shown that in a bilayer excitonic superconductor dissipative losses emerge under transmission of the current from the source to the load. These losses are proportional to the square of the interlayer tunneling amplitude and are independent of the value of the input current. The case of a quantum Hall bilayer is considered. The bilayer may work as a transmission line if the input current exceeds a certain critical value. An input current higher than the critical one induces Josephson vortices in the bilayer. The difference in electrochemical potentials is required to feed the load and it forces Josephson vortices to move. The state becomes non-stationary which leads to dissipation.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Among the phenomena that demonstrate bilayer electron systems in semiconductor heterostructures considerable attention is given to the superfluidity of electron–hole (e–h) pairs with components belonging to different layers (see, for instance, [1]). A flow of electron–hole pairs in the bilayer is equivalent to two oppositely directed electrical currents in the layers. Therefore, the superflow of such pairs is a kind of superconductivity. For the first time the effect was considered in [2, 3] for bilayers where one layer is of electron-type conductivity and the other layer of hole-type conductivity (electron–hole bilayers). The next important step was the prediction of the e–h superfluidity for the systems where the conductivity of both layers is of the same type [4, 5]. In that case the bilayer should be subjected to a strong magnetic field perpendicular to the layers. If the total filling factor of the Landau levels is $\nu = 1$ the number of electrons in one layer coincides with the number of holes in the other layer (the empty states in the lowest Landau level play the role of holes) and the Coulomb attraction between electrons and holes results in the formation of bound pairs. Note that the description of electron–hole pairing in electron–electron bilayers in a quantized magnetic field is close to that developed earlier for the quantum Hall electron–hole bilayers [6–8].

The prediction [4, 5] has inspired a considerable increase of interest in the study of this phenomenon, both theoretically [9–17] and experimentally [18–26]. The results of recent experimental investigations of quantum Hall bilayers support the idea of superfluidity of e–h pairs in these systems. In particular, in the counterflow experiments a huge increase of longitudinal conductivity was observed [18–20]. Other important observations are a large low bias tunnel conductivity [21], which also takes place in bilayers with a large imbalance of filling factors [26], the Goldstone collective mode [22], the quantized Hall drag between the layers [23], the interlayer drag [24] and the interlayer critical supercurrent [25] in the Corbino disc geometry. The low temperature properties of optically generated indirect excitons in bilayers in zero magnetic field were also studied experimentally [27, 28], and specific features in the photoluminescence spectra accounting for the Bose–Einstein condensation of electron–hole pairs have been observed. A recent important contribution to this topic is connected with the idea of using two graphene layers separated
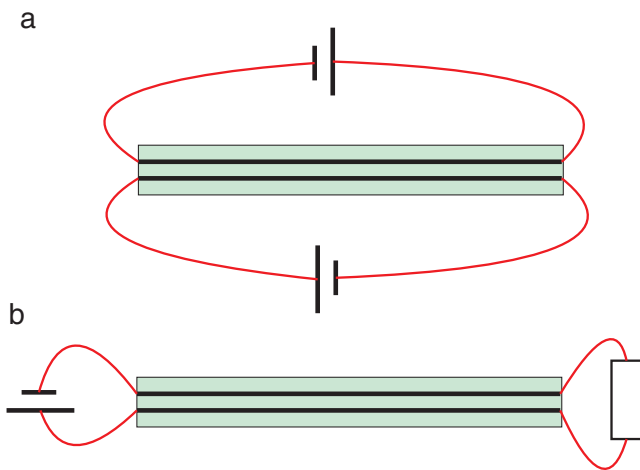
Figure 1. The load (a) and the loop (b) set-ups.

Figure 2. A circuit with a stack of bilayers (a) and an equivalent circuit with shunted bilayers (b).

by a dielectric layer [29–32] instead of GaAs heterostructures. It is expected [29] that in graphene systems in zero magnetic field superconductivity of e–h pairs may take place at rather high temperatures.

A superfluid state of electron–hole pairs in quantum Hall bilayers can be considered as a state with spontaneous interlayer phase coherence between the electrons. The coherent phase φ is the phase of the order parameter for the electron–hole pairing and the gradient of the phase φ determines the value of the supercurrent in the layer (planar supercurrent).

The bilayer system in the counterflow set-up [18–20] can be used as a double-wire circuit that transmits the current from the source to the load. If the transmission is provided by superfluid e–h pairs one can expect that such a line will work as a usual superconducting transmission line. Nevertheless, genuine superconductivity was not achieved in the counterflow experiments. A finite resistance can be accounted for a disorder that results in a finite concentration of planar vortices in the bilayer [33, 34]. But there is another principal circumstance that may forbid genuine superconductivity in such a transmission line. To support nonzero current in the load circuit a difference in electrochemical potentials between the layers is required. This difference results in a temporal dependence of the phase φ. The state with the spontaneous interlayer phase coherence emerges if the layers are situated rather close to each other (at a distance less than the magnetic length). Therefore, nonzero interlayer tunneling is always present in real physical systems. At nonzero interlayer tunneling amplitude the difference in electrochemical potentials results in the appearance of an ac Josephson current and the state becomes a non-stationary one. As was shown in [35] dissipative losses emerge in the non-stationary state. The situation is different for the loop and for the load geometry [36]. The steady state is not possible in the loop geometry in figure 1(b) (which corresponds to the transmission line [18–20]) but it can be realized in the load geometry shown in figure 1(a) under the condition that the difference in the electrochemical potential of the layers is tuned to zero.
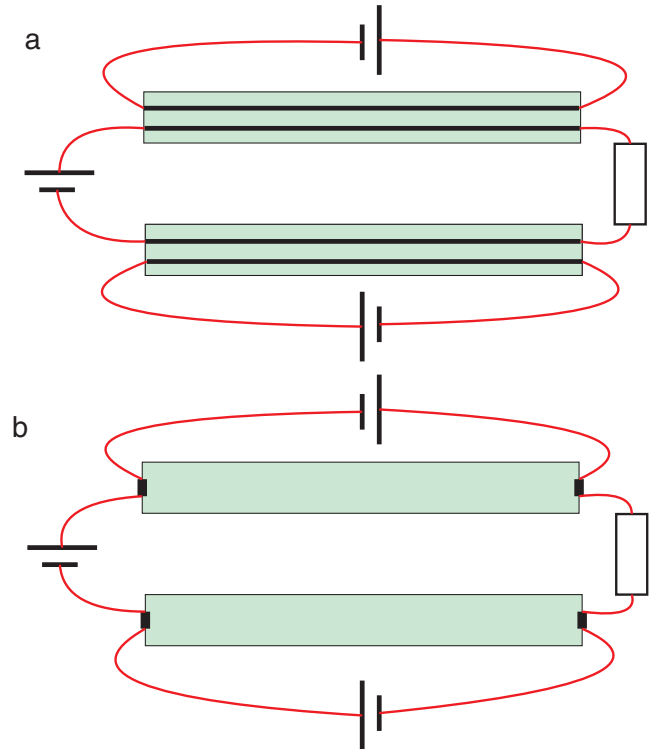
The load geometry [36] is an appropriate set-up for the observation of the superfluidity of e–h pairs but it cannot be used for the transmission of the current from the source to the load. In more complicated circuits, e.g. made of a stack of bilayers, non-dissipative transmission of the energy is not possible as well. Indeed, a bilayer with zero difference of electrochemical potentials can be shunted at both ends without any impact on electrical currents and voltages in external circuits. It means that the removal of such a bilayer from the circuit cannot change its working parameters, including the power of losses. An example of such a shunting is shown in figure 2. The only set-up where superfluid properties of e–h pairs may lead to a lowering of dissipation is the system with nonzero difference in electrochemical potentials between the layers. While such a system is not free from dissipation, the losses can be exponentially small. Then, the behavior of the system is practically indistinguished from one of genuine superconductors.

In this paper we consider dissipative processes in the bilayer in the loop geometry and find the conditions at which the losses are small. The state with ac Josephson and planar supercurrents that emerges in the bilayer can be described as a moving chain of Josephson vortices. Under such a motion two mechanisms of dissipation come into play. They are the second viscosity that results in dissipation in the non-stationary regime and Joule losses caused by ac electrical fields that emerge due to spatial and temporal variation of the interlayer voltage.

In section 2 starting from the microscopic Hamiltonian and using the BCS-like many-particle wavefunction we derive the stationary continuity equation. In section 3 stationary

vortex states are considered and the problem of the lower critical current is addressed. In section 4 the state with moving vortices is investigated and the power of losses caused by such a motion is computed. We find the power of losses is proportional to the square of the tunneling amplitude and depends nonlinearly on the resistance of the load.

2. The model

Let us consider a bilayer electron system subjected to a strong magnetic field B perpendicular to the layers. The filling factors of the layers satisfy the condition $v_1 + v_2 = 1$, where $v_i = 2\pi \ell^2 n_i$, n_i, are the electron densities in the layers and $\ell = \sqrt{\hbar c/eB}$ is the magnetic length. Implying the energy gap between the Landau levels $\hbar\omega_c$ (ω_c is the cyclotron frequency) is larger than the Coulomb energy $e^2/\varepsilon\ell$ (ε is the dielectric constant) we use the lowest Landau level approximation. In this approximation the Hamiltonian of the system has the form

$$H = -t \sum_X (a_{1X}^\dagger a_{2X} + \text{H.c.}) + \frac{1}{2S} \sum_{n,n'=1,2} \sum_{\mathbf{q}} V_{n,n'}(\mathbf{q})$$
$$\times [\rho_n(\mathbf{q})\rho_{n'}(-\mathbf{q}) - \delta_{nn'} e^{-\frac{q^2\ell^2}{2}} \rho_n(0)]. \quad (1)$$

Here $a_{nX}^\dagger$, a_{nX} are the creation and annihilation operators for the electrons in the nth layer, X is the guiding center of the orbit, S is the area of the layer, t is the interlayer tunneling amplitude, $\rho_n(\mathbf{q}) = \sum_X a_{n,X+q_y\ell^2/2}^\dagger a_{n,X-q_y\ell^2/2} e^{iq_x X - q^2\ell^2/4}$ is the operator for the Fourier component of the electron density and $V_{n,n'}(\mathbf{q}) = (2\pi e^2/\varepsilon q) \exp(-qd|n-n'|)$ is the Fourier component of the Coulomb potential (d is the distance between the layers). In what follows we consider the bilayer of a rectangular shape ($S = L_x \times L_y$) with the planar currents directed parallel to the x axis.

The state with the interlayer phase coherence is described by the wavefunction

$$|\Psi\rangle = \prod_X (u a_{1X}^\dagger + v a_{2X}^\dagger)|0\rangle, \quad (2)$$

where the coefficients u and v satisfy the condition $|u|^2 + |v|^2 = 1$. The physical meaning of this function and its applicability for a description of the superfluid state of e–h pairs in quantum Hall bilayers was discussed in [9–11] and in a number of further papers. Here we just recall the main points. The function (2) can be presented in another equivalent form:

$$|\Psi\rangle = \prod_X (u + v a_{2X}^\dagger h_{1X}^\dagger)|\text{vac}\rangle, \quad (3)$$

where $h_n^\dagger = a_n$ is the creation operator for the hole and the vacuum state $|\text{vac}\rangle$ is the state in which layer 1 is fully occupied and layer 2 is empty ($v_1 = 1$ and $v_2 = 0$). It was shown [8] that the function (2) and (3) corresponds to the ground state of the Hamiltonian (1) in the limit $d/\ell \to 0$. One can see that the function (3) is the standard BCS form for the exciton condensate. The exciton is thought of as an electron in layer 2 bound to a hole in layer 1. One can show (see, for instance, [37]) that in the first quantized language the function (2) and (3) is reduced to the (111) Halperin

wavefunction. At finite d/ℓ the state (2) and (3) is not the exact ground state. Nevertheless, numerical studies show (see, e.g., [38]) that at $d/l < 0.5$ the overlap with the exact ground state is close to 100%. For the system with given filling factors of the layers the u–v coefficients in (2) and (3) are $u = \sqrt{v_1}$ and $v = e^{i\varphi}\sqrt{v_2}$. In the absence of the interlayer tunneling the energy of the state (2) and (3) does not depend on the phase φ. In the case of spatially dependent phase $\varphi = \varphi(X)$ the function (2) and (3) describes the state with nonzero counterflow electrical currents in the layers (see [9]).

The order parameter for the exciton condensate at $T = 0$ is

$$\Delta(X) = \langle\Psi|a_{2X}h_{1X}|\Psi\rangle = u_X^* v_X = \Delta_0 e^{i\varphi(X)}, \quad (4)$$

where $\Delta_0 = \sqrt{v_1(1-v_1)}$. Due to the absence of the kinetic energy of carriers in the Landau level the self-consistency equation for the order parameter at $T = 0$ is reduced to an algebraic equation. At nonzero temperature the equation for the order parameter takes a self-consistent form [6, 39] from which the temperature dependence of the order parameter and the mean-field critical temperature can be obtained. But due to the two-dimensional nature of the bilayer excitons the temperature of transition into the superfluid state is not the mean-field critical temperature, but the Berezinskii–Kosterlitz–Thouless transition temperature $T_{\text{BKT}} \approx \pi\rho_s/2$, see [9] (the definition of the superfluid stiffness ρ_s is given below).

If the gradient of the phase is small in comparison to the inverse magnetic length, the energy of the system $E = \langle\Psi|H|\Psi\rangle$ can be written in the continuous approximation

$$E = E_0 + \int d^2r \left[\frac{1}{2}\rho_s \left(\frac{d\varphi}{dx}\right)^2 - \frac{\tilde{t}}{2\pi\ell^2}(\cos\varphi - 1) \right], \quad (5)$$

where E_0 is the energy of the ground state, $\tilde{t} = 2t\Delta_0$ is the tunneling energy and

$$\rho_s = \Delta_0^2 \frac{e^2}{4\pi\varepsilon\ell} \left[\sqrt{\frac{\pi}{2}} \exp\left(\frac{d^2}{2\ell^2}\right) \text{erfc}\left(\frac{d}{\ell\sqrt{2}}\right) \right.$$
$$\left. \times \left(1 + \frac{d^2}{\ell^2}\right) - \frac{d}{\ell} \right] \quad (6)$$

is the energy parameter called the superfluid stiffness. The superfluid stiffness can also be presented in a more familiar form $\rho_s = \hbar^2 n_s/m_B$, where n_s is the superfluid density of the pairs and m_B is the magnetic mass of the pair (see, for instance, [14]). One can see from (5) that at nonzero tunneling amplitude the ground state corresponds to the phase $\varphi = 0$. But it does not mean the fixation of the phase and the absence of electrical currents. It means that the planar current should be accompanied by the interlayer current.

Varying the energy (5) with respect to the phase and equating the result to zero we obtain the equation

$$-\frac{e\rho_s}{\hbar} \frac{d^2\varphi}{dx^2} + \frac{e}{\hbar} \frac{\tilde{t}}{2\pi\ell^2} \sin\varphi = 0. \quad (7)$$

The first term in equation (7) is the 2D divergence of the density of the planar supercurrent:

$$j_1 = -j_2 = -\frac{e\rho_s}{\hbar} \frac{d\varphi}{dx}. \quad (8)$$

The expression for the planar current can be obtained from the gauge invariance arguments (see [15, 40]).

Thus, equation (7) is nothing more than the stationary continuity equation

$$\frac{\mathrm{d}j_1}{\mathrm{d}x} + I_{1\to2} = 0, \qquad (9)$$

where

$$I_{1\to2} = \frac{e}{\hbar}\frac{\tilde{t}}{2\pi\ell^2}\sin\varphi \qquad (10)$$

is the density of the interlayer supercurrent flowing from layer 1 to layer 2.

The coherent electrical current between the layers corresponds to nonzero φ, in a close analogy to the Josephson effect between two bulk superconductors. The difference is that φ is the phase of a given condensate, but not the phase difference for two condensates.

Equation (7) can be presented in the form

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{1}{\lambda^2}\sin\varphi, \qquad (11)$$

where $\lambda = \ell\sqrt{2\pi\rho_s/\tilde{t}}$ is the Josephson length. As follows from equation (11), the gradient of the phase is of the order of λ^{-1}. Therefore, the continuity approximation (5) requires $\lambda \gg \ell$. This inequality is fulfilled if the tunneling amplitude t is much smaller than the Coulomb energy $e^2/\varepsilon\ell$ and the filling factor ν is not very close to zero.

3. Stationary vortex state and lower input critical current

The problem of the lower critical current was already discussed in a number of papers [12, 15, 41]. Here we revisit this question again with the aim of giving a more accurate definition of the lower critical current and to clarify some discrepancies in [12, 15, 41].

Let us consider the following situation. A fixed input current from a source is entered into one layer in a given (source) end of the system. There is no difference between the electrochemical potentials of the layers and the stationary state is realized. The current withdrawn from the adjacent layer at the same end is equal to the input current. The values of the output and input currents at the opposite (load) end are also equal to each other but, in general, they may differ from the currents at the source end. For instance, such a situation corresponds to the load geometry [36] (then all input and output currents are equal to one another). In the loop geometry the stationary state can be realized if the load has zero resistance (superconducting load), or if an additional source in the load circuit provides zero difference in electrochemical potentials between the layers.

Depending on the values of the input currents two qualitatively different stationary current patterns can emerge: either the planar and interlayer currents are nonzero in the whole bilayer, or the currents decrease exponentially with the distance from the ends and there is no current in the internal part of the bilayer. We define the lower critical input current as

the current at which the switching between these two regimes occurs. If the input currents at both ends are lower than the critical one the second pattern is realized and the interior part of the bilayer is not involved in the transmission of the current (the cutting of the bilayer does not change currents in the outer circuits). If one or both input currents exceed j_c the first pattern emerges.

Let us switch to the quantitative analysis. Equation (11) coincides in form with the equation of motion for a nonlinear pendulum:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \omega_0^2\sin\varphi, \qquad (12)$$

where ω_0 is the resonant frequency of the pendulum and φ is the angle coordinate counted from the unstable equilibrium point. The formal coincidence of equations (11) and (12) allows us to describe the stationary states in the bilayer basing on the known behavior of a nonlinear pendulum.

Depending on the energy, the pendulum oscillates, completes a full revolution in infinite time, or rotates. The maximum angular velocity ω_m of the pendulum (velocity at $\varphi = \pi$) is proportional to the square root of the energy and it increases if one switches from the oscillating to the rotating regime. For the full revolution in infinite time the maximum angular velocity $\omega_m = 2\omega_0$.

The time-dependent angular velocity in the pendulum problem corresponds to the space-dependent planar supercurrent in the bilayer problem. The counterpart of the full revolution regime is a state with one Josephson vortex. In this state the planar and the interlayer supercurrents are given by the expressions

$$j_1 = -j_2 = j_c\operatorname{sech}\left(\frac{x - x_0}{\lambda}\right),$$

$$I_{1\to2} = \frac{j_c}{\lambda}\operatorname{sech}\left(\frac{x - x_0}{\lambda}\right)\tanh\left(\frac{x - x_0}{\lambda}\right), \qquad (13)$$

where x_0 is the vortex center and

$$j_c = \frac{e\rho_s}{\hbar}\frac{2}{\lambda} \qquad (14)$$

is the maximum value of the planar supercurrent in the single vortex state (it is reached at the center of the vortex). Note that equation (14) can be obtained directly from equation (8) under accounting for the correspondences $(\mathrm{d}\varphi/\mathrm{d}x)_{\max} \Leftrightarrow \omega_m = 2\omega_0$ and $\omega_0 \Leftrightarrow 1/\lambda$.

The counterpart of the rotation regime is the state with many equally distanced Josephson vortices with the same sign of vorticity. The supercurrents in this state are

$$j_1 = -j_2 = \frac{j_c}{\sqrt{\eta}}dn\left(\frac{x}{\lambda\sqrt{\eta}}, \eta\right),$$

$$I_{1\to2} = \frac{j_c}{\lambda}sn\left(\frac{x}{\lambda\sqrt{\eta}}, \eta\right)cn\left(\frac{x}{\lambda\sqrt{\eta}}, \eta\right). \qquad (15)$$

The analog of the oscillating motion of the pendulum is the multivortex state with vortices of alternating vorticity:

$$j_1 = -j_2 = j_c\sqrt{\eta}cn\left(\frac{x}{\lambda}, \eta\right),$$

$$I_{1\to2} = \frac{j_c\sqrt{\eta}}{\lambda}dn\left(\frac{x}{\lambda}, \eta\right)sn\left(\frac{x}{\lambda}, \eta\right). \qquad (16)$$

In equations (15) and (16) $dn(u, \eta)$, $cn(u, \eta)$ and $sn(u, \eta)$ are the Jacobi elliptic functions and the parameter η belongs to the interval (0, 1).

The boundary condition for the bilayer problem is the condition on the gradients of the phase. This condition alone does not determine a unique current state. The system chooses the state that, at given input currents, has the lowest energy. Direct calculation shows that the energies of the multivortex state (15) and (16) are both higher than the energy of the single vortex state (13). It is an obvious result since the energy of any multivortex state is proportional to the number of vortices.

Here we should emphasize the difference between the pendulum problem and the bilayer problem. The energy of the pendulum is the integral of motion of equation (12), while the bilayer energy (5) is not the first integral of equation (11). Therefore, the analogy between these two problems fails if one analyzes the energies of different states.

One can see that the planar supercurrent may exceed j_c only in the state (15). Therefore, if the input current j_{in} is higher than j_c the multivortex state (15) is realized. The parameter η is determined by the conditional minimum of energy and it is equal to $\eta = (j_c/j_{in})^2$. The density of the Josephson vortices increases under an increase of the input current.

If the input current is smaller than j_c the state with one incomplete vortex centered outside the bilayer satisfies the boundary condition and corresponds to the minimum of energy. Thus, the state realized at $j_{in} \leqslant j_c$ is the state with zero supercurrents inside the bilayer.

The vortex state with alternating vorticities (16) is not realized in the bilayers. It may satisfy the boundary conditions only at $j_{in} < j_c$, but its energy is larger than the energy of the state with one vortex. It contradicts the conclusion of [41], but in [41] the first integral of equation (11) was incorrectly identified with the energy.

Thus, the quantity (14) is just the lower input critical current defined above.

At $j_{in} \gg j_c$ the overlapping between the vortices is large and the current pattern is approximated by a sum of the large dc part and the small harmonic ac part:

$$\varphi \approx \pi - kx - A_\varphi \sin kx,$$

$$j_1 = -j_2 \approx \frac{j_{in}}{1 + A_\varphi}(1 + A_\varphi \cos kx), \qquad (17)$$

$$I_{1 \to 2} \approx \frac{e}{\hbar} \frac{\tilde{t}}{2\pi \ell^2} \sin kx,$$

where

$$k = \frac{\pi j_{in}}{j_c \lambda K\left(\frac{j_c^2}{j_{in}^2}\right)} \approx \frac{2j_{in}}{j_c \lambda}, \qquad A_\varphi = \frac{j_c^2}{4j_{in}^2}.$$

Here $K(\eta)$ is the complete elliptic integral of the first kind.

To conclude this section we note that at small input current (lower than j_c) the tunneling provides shunting of the bilayer at both ends. It means that the load geometry experiments proposed in [36] for the observation of e–h superfluidity should be done at large currents.

4. Vortex motion and dissipation

As was already discussed in section 1, the stationary state cannot be realized in the bilayer transmission line with nonzero load resistance. But even in the case of non-stationary currents inside a bilayer the currents in external circuits may remain stationary. The latter occurs if the contacts between the layers and the external wires average the input and output currents.

In such a situation the term 'lower critical current' slightly changes its meaning. The average current depends on the density of vortices and can be smaller than j_c. But an input current higher than the critical one is required at the beginning of the process to induce a chain of moving vortices.

To study the non-stationary vortex state we will use the set of equations for the phase of the order parameter φ and for the local difference of the electrochemical potentials between the layers eV, where V is the local interlayer voltage. The first equation comes from the non-stationary continuity equation:

$$e\frac{\partial \tilde{n}}{\partial t} + \frac{\partial j_1^s}{\partial x} + I_{1 \to 2} + \frac{\partial j_1^n}{\partial x} = 0. \qquad (18)$$

Here $\tilde{n}$ is the excess local density of electron–hole pairs. It is connected with the local voltage by the capacitor equation

$$\frac{e\tilde{n}}{V} = C, \qquad (19)$$

where

$$C = \frac{\varepsilon}{4\pi d} \frac{1}{1 - \frac{\ell}{d}\sqrt{\frac{\pi}{2}}\left[1 - \exp\left(\frac{d^2}{2\ell^2}\right)\text{erfc}\left(\frac{d}{\ell\sqrt{2}}\right)\right]} \qquad (20)$$

is the effective capacity of the bilayer system per unit area. The effective capacity takes into account the exchange interaction between the layers and differs from the classical capacity by an additional factor (the second factor in equation (20)). It can be obtained from the dependence of the energy on the filling factor imbalance (see, e.g., [39]). The superscripts 's' and 'n' in equation (18) stand for the planar supercurrent and the planar current of quasiparticles. The current of quasiparticles is taken into account in equation (18) because it is induced by ac planar electrical fields that emerge in the non-stationary state. These fields are equal to $E_1 = -E_2 = -(1/2)\partial V/\partial x$ and the quasiparticle current can be presented in the form $j_1^{(n)} = -(\sigma_n/2)\partial V/\partial x$, where σ_n is the quasiparticle conductivity. We do not take into account the normal component of the interlayer current because the normal tunneling is suppressed due to the discreteness of the Landau levels.

The second equation for φ and V can be derived from the equations of superfluid hydrodynamics [42], namely from the equation for the superfluid velocity with dissipative terms. The linearized version of this equation is

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \mu_m + \nabla \left[\zeta_3 \nabla \cdot n_s(\mathbf{v}_s - \mathbf{v}_n) + \zeta_4 \nabla \cdot \mathbf{v}_n\right], \qquad (21)$$

where μ_m is the chemical potential per unit mass $\mu_m = \mu/m$ (m is the mass of the Bose particle), n_s is the superfluid density, $\mathbf{v}_n$ is the velocity of the normal component, and ζ_3 and ζ_4 are the second viscosity parameters. To apply equation (21) for

the description of the bilayer electron–hole superfluidity one should replace the divergence of the superfluid flow in (21) with the divergence of the planar supercurrent plus the tunnel supercurrent: $\nabla(n_s \mathbf{v}_s) \Rightarrow (1/e)(\partial j_1^s/\partial x + I_{1 \to 2})$, and substitute the difference of the electrochemical potentials instead of the chemical potential $\mu \Rightarrow eV$. Using equations (10) and (8), and the expression for the superfluid velocity $\mathbf{v}_s = -\hbar \nabla \varphi / m$ (the negative sign is because the phase φ describes the coherence between the electrons), we obtain

$$\nabla \left(\hbar \frac{\partial \varphi}{\partial t} \right) = \nabla \left[eV + \zeta_3 \frac{m}{2\pi \ell^2 \hbar} \left(2\pi \ell^2 \rho_s \frac{\partial^2 \varphi}{\partial x^2} - \tilde{t} \sin \varphi \right) \right. $$
$$\left. + \mathrm{O}(v_n) \right]. \tag{22}$$

The quantity $\mathrm{O}(v_n)$ in equation (22) stands for the term caused by the motion of the normal component. Below we will neglect the $\mathrm{O}(v_n)$ term. One can show that this term yields the correction to the power of losses quadratic in the dissipative coefficients, while the leading term is linear in these coefficients. Since we are interested in the vortex solution, the gradients in (22) can be omitted.

Thus, we obtain the following set of equations for φ and V:

$$C \frac{\partial V}{\partial t} = \frac{e}{\hbar} \rho_s \frac{\partial^2 \varphi}{\partial x^2} - \frac{e}{\hbar} \frac{\tilde{t}}{2\pi \ell^2} \sin \varphi + \frac{\sigma_n}{2} \frac{\partial^2 V}{\partial x^2}, \tag{23}$$

$$\hbar \frac{\partial \varphi}{\partial t} = eV + \alpha_\varphi \left(2\pi \ell^2 \rho_s \frac{\partial^2 \varphi}{\partial x^2} - \tilde{t} \sin \varphi \right). \tag{24}$$

The coefficient $\alpha_\varphi = \zeta_3 m / 2\pi \ell^2 \hbar$ in equation (24) is the second superfluid viscosity expressed in dimensionless units (with $m = m_B$, the magnetic mass of the e–h pair). Equations (23) and (24) contain two terms responsible for the dissipation. The term proportional to σ_n describes the Joule losses caused by ac quasiparticle currents and the term proportional to α_φ describes the losses caused by the second viscosity. The equations with the same form of dissipative terms were used in [43]. Equation (24) was also obtained in [44] for the electron–hole bilayers in zero magnetic field in the 'dirty' limit.

The channels of dissipation caused by the second viscosity and quasiparticle currents are present in any superfluid system, but they do not result automatically in dissipative losses. One can see that in the stationary regime the dissipative terms in equations (23) and (24) equal zero. The losses appear in the presence of additional factors. For the bilayer transmission line the main factor is non-stationarity of the interlayer and the planar supercurrents.

As was shown in section 3, the input current may induce Josephson vortices in the bilayer system. At nonzero difference of the electrochemical potentials the vortices begin to move from the source to the load. Neglecting for a moment the dissipative terms we obtain from equations (23) and (24) the following equation for the phase:

$$\frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\partial^2 \varphi}{\partial \tau^2} = \sin \varphi, \tag{25}$$

where the dimensionless variables $\xi = x/\lambda$ and $\tau = t/\tau_0$ ($\tau_0 = (\hbar \ell / e) \sqrt{2\pi C / \tilde{t}}$) are used. Equation (25) has the solution $\varphi(\xi, \tau) = \phi(\xi - \beta \tau)$, where $\beta \neq 1$ and $\phi(u)$ satisfy the equation

$$\frac{\partial^2 \phi}{\partial u^2} = \frac{1}{1 - \beta^2} \sin \phi \tag{26}$$

which up to the notations coincides with equation (11). Using the results of section 3 we obtain the following expression for the planar current and for the interlayer voltage:

$$j_1^s(x, t) = \frac{2e}{\hbar} \rho_s k \, dn(kx - \omega t, \eta), \tag{27}$$

$$V(x, t) = \frac{2\hbar \omega}{e} dn(kx - \omega t, \eta), \tag{28}$$

where

$$k = \frac{1}{\lambda \sqrt{\eta |1 - \beta^2|}}, \qquad \omega = \frac{\beta}{\tau_0 \sqrt{\eta |1 - \beta^2|}}. \tag{29}$$

Equation (27) describes the moving vortex lattice. The velocity of motion is

$$u_v = \frac{\omega}{k} = \frac{e^2}{\hbar^2} \rho_s \frac{\langle V \rangle}{\langle j_1^s \rangle} = \frac{e^2 \rho_s R L_y}{\hbar^2}. \tag{30}$$

Here and below $\langle \cdots \rangle$ means the time average. In equation (30) R is the resistance of the load (we imply Ohm's law $\langle V \rangle = R L_y \langle j_1^s \rangle$ for the load circuit). One can see that the vortex velocity is proportional to the load resistance and does not depend on the input current. The parameter $\beta = u_v \tau_0 / \lambda$ is also proportional to the load resistance.

Let us switch to the case of small but nonzero dissipation. The power of losses can be obtained from the common expression for the Joule losses:

$$\Delta P = L_y \left\langle \int_0^{L_x} dx \left[I_{1 \to 2} V + (j_1^s + j_1^n) \left(-\frac{\partial V}{\partial x} \right) \right] \right\rangle. \tag{31}$$

Integrating (31) by parts and taking into account the continuity equation (18) we obtain the obvious result that the power of losses is the input power minus the output power:

$$\Delta P = (j_{in} - j_{out}) V_0 L_y. \tag{32}$$

Here j_{out} is the output planar current and V_0 is the voltage in the output circuit (which coincides with the voltage applied to the bilayer system at the input end). In deriving (32) we take into account that $\langle V(\partial V / \partial t) \rangle = 0$ for any function $V(x, t)$ periodic in t.

The difference between the input and the output average currents emerges if the average value of the interlayer current $\bar{I} = \langle I_{1 \to 2} \rangle$ differs from zero. In what follows we consider the case where the power of losses is much smaller than the input power. In this case one can neglect the dependence of $\bar{I}$ on x and approximate the difference $j_{in} - j_{out}$ as $j_{in} - j_{out} = L_x \bar{I}$. Then the power of losses is

$$\Delta P = \bar{I} V_0 S. \tag{33}$$

The quantity $\bar{I}$ can be found from the solutions of equations (23) and (24). Here we consider the case of large input current $j_{\rm in} \gg j_{\rm c}$. In this case one can seek for a solution of equations (23) and (24) in the following form:

$$\varphi(x,t) = \omega t - kx + A_\varphi \sin(\omega t - kx) + B_\varphi \cos(\omega t - kx), \quad (34)$$

$$V(x,t) = V_0 \left(1 + A_V \sin(\omega t - kx) + B_V \cos(\omega t - kx)\right), \quad (35)$$

where $\omega = eV_0/\hbar$ and the quantity k is connected with the planar supercurrent by the relation $\langle j_1^{\rm s} \rangle = e\rho_{\rm s} k/\hbar$. Substituting equations (34) and (35) into equations (23) and (24) and neglecting the terms quadratic in $\tilde{t}$ we obtain the coefficients A_φ, B_φ, A_V and B_V in the leading order in $\tilde{t}$:

$$A_\varphi = -\frac{\tilde{t}}{2\pi \ell^2 k^2 \rho_{\rm s}} \left\{ (1 - \beta^2 + 2\pi \ell^2 k^2 \alpha_\varphi \tilde{\sigma})(1 + 2\pi \ell^2 k^2 \alpha_\varphi \tilde{\sigma}) \right.$$
$$\left. + \alpha_\varphi \frac{V_0}{V_C} \left(\alpha_\varphi \frac{V_0}{V_C} + \tilde{\sigma} \frac{V_0}{V_\rho} \right) \right\} \{D\}^{-1}, \quad (36)$$

$$B_\varphi = \frac{\tilde{t}}{2\pi \ell^2 k^2 \rho_{\rm s}} \frac{\tilde{\sigma} \frac{V_0}{V_\rho} \left(1 + 2\pi \ell^2 k^2 \alpha_\varphi \tilde{\sigma}\right) + \alpha_\varphi \frac{V_0}{V_C} \beta^2}{D}, \quad (37)$$

$$A_V = -\frac{\tilde{t}}{2\pi \ell^2 k^2 \rho_{\rm s}} \frac{\alpha_\varphi \frac{V_0}{V_C} + \tilde{\sigma} \frac{V_0}{V_\rho}}{D}, \quad (38)$$

$$B_V = -\frac{\tilde{t}}{2\pi \ell^2 k^2 \rho_{\rm s}} \frac{1 + 2\pi \ell^2 k^2 \alpha_\varphi \tilde{\sigma} - \beta^2}{D}, \quad (39)$$

where

$$D = (1 - \beta^2 + 2\pi \ell^2 k^2 \alpha_\varphi \tilde{\sigma})^2 + \left(\alpha_\varphi \frac{V_0}{V_C} + \tilde{\sigma} \frac{V_0}{V_\rho} \right)^2, \quad (40)$$

and the following notations are used:

$$\beta^2 = \frac{CV_0^2}{\rho_{\rm s} k^2}, \qquad V_C = \frac{e}{2\pi \ell^2 C}, \qquad V_\rho = \frac{\rho_{\rm s}}{e},$$
$$\tilde{\sigma} = \frac{\sigma_{\rm n} \hbar}{2e^2}. \quad (41)$$

Note the equivalence of β in (41) and the definition of β given above.

One can see that at $\tilde{t} = 0$ all the coefficients A_φ, B_φ, A_V and B_V are equal to zero, which reflects the absence of vortices in the bilayers with zero interlayer tunneling.

Using equations (10) and (34) one finds the average value of the interlayer current:

$$\bar{I} = \frac{e\tilde{t}}{2\pi \ell^2 \hbar} \langle \sin \varphi \rangle \approx \frac{e\tilde{t}}{4\pi \ell^2 \hbar} B_\varphi. \quad (42)$$

Since B_φ is proportional to $\tilde{t}$ the average interlayer current is proportional to the square of the matrix element of the interlayer tunneling. The higher-order corrections to B_φ yield the contributions $\propto \tilde{t}^3$ to $\bar{I}$. Therefore, for obtaining the power of losses in the leading order it is enough to use the linear in $\tilde{t}$ approximation for the phase.

Using equations (42) and (37) we obtain the following expression for the power of losses:

$$\frac{\Delta P}{S} = \frac{\tilde{t}^2}{4\pi \ell^2 \hbar} F\left(\frac{R}{R_0}\right) \quad (43)$$

where

$$F\left(\frac{R}{R_0}\right) = \left\{ \tilde{\sigma} \frac{V_C}{V_\rho} \left(\frac{R}{R_0}\right)^2 + \alpha_\varphi \left(\frac{R}{R_0}\right)^4 \right\}$$
$$\times \left\{ \left(1 - \left(\frac{R}{R_0}\right)^2 + 2\pi \alpha_\varphi \tilde{\sigma} \left(\frac{\bar{j}}{j_0}\right)^2 \right)^2 \right.$$
$$\left. + 2\pi \left(\frac{R}{R_0}\right)^2 \left(\frac{\bar{j}}{j_0}\right)^2 \frac{V_\rho}{V_C} \left(\alpha_\varphi + \tilde{\sigma} \frac{V_C}{V_\rho}\right)^2 \right\}^{-1}. \quad (44)$$

Here $j_0 = e\rho_{\rm s}/\hbar \ell$ ($j_0 \gg j_{\rm c}$) is the current of the order of the higher critical current [15] (the current above which the superfluid state is destroyed). The quantity $R_0 = \hbar/eL_y \sqrt{\rho_{\rm s} C}$ is the resonant load resistance. One can see that dissipative losses increase considerably as R approaches to R_0 (the condition $R = R_0$ is equivalent to $\beta = 1$). We emphasize that the approximation solution (34) and the answer (44) are not valid in a resonant case and at $R \approx R_0$ they describe the situation only qualitatively.

Under obtaining equation (33) we take into account the relation $\langle j_1^{\rm s} \rangle = \bar{j} = \frac{e}{\hbar} \rho_{\rm s} k$ and Ohm's law $V_0 \approx R_l L_y \bar{j}$ in the load circuit. In equation (44) the terms in the denominator quadratic in dissipative coefficients are omitted.

Thus we conclude that the power of losses is proportional to the square of the tunneling amplitude and depends nonlinearly on the load resistance. At small load resistance $R \ll R_0$ the dissipation is connected in the main part with the conductivity of quasiparticles. It is proportional to the square of the load resistance:

$$\frac{\Delta P}{S} \approx \frac{\tilde{t}^2}{4\pi \ell^2 \hbar} \tilde{\sigma} \left(\frac{R}{R_0}\right)^2 \frac{V_C}{V_\rho} \quad (45)$$

and vanishes at $R = 0$. At large load resistance $R \gg R_0$ the main contribution to the power of losses comes from the second viscosity. The power of losses at large R approaches the constant value

$$\frac{\Delta P}{S} \approx \frac{\tilde{t}^2}{4\pi \ell^2 \hbar} \alpha_\varphi. \quad (46)$$

The resonant resistance R_0 does not depend on the tunneling amplitude but it can be tuned by a change of d/ℓ (it is the increasing function of that parameter). At $d/\ell = 1$ the resonant resistance is approximated as

$$R_0 \approx \frac{h}{e^2} \frac{2}{\Delta_0} \frac{\ell}{L_y}. \quad (47)$$

At low load resistance the 'bottleneck' of the load circuit is the contacts where the interlayer phase coherence is broken. We evaluate from (47) that at longitudinal resistivity of the layer $\rho_{xx} = 1$ kΩ the resistance of the arm $R \sim R_0$ corresponds to the length of contacts $\sim 10^2 \ell \approx 1~\mu$m. Since it is a rather small length we conclude that the experimental situation corresponds most probably to the case $R \gg R_0$.

In conclusion we note that in the non-resonant regime the power of losses does not depend on the input current. Since the input power is proportional to the input current, the efficiency factor increases under an increase of the input current.

5. Conclusion

We have shown that in the bilayer system with superfluid electron–hole pairs the dissipation appears under the transmission of the current from the source to the load. The effect is connected to that the electrical currents inside the bilayer become non-stationary at nonzero difference of the electrochemical potentials between the layers.

The non-stationary state can be interpreted as the state with moving Josephson vortices. It is well known that in type-II superconductors the motion of quantum vortices results in dissipation, but the latter can be eliminated by pinning of the vortices. In this connection one can think that the pinning may also suppress the dissipation in the bilayers. But it is not true. The difference between type-II superconductors and the bilayers is the following. In type-II superconductors nonzero voltage along the direction of the supercurrent emerges due to the vortex motion. In the bilayers nonzero interlayer voltage is required to support electrical current in the load circuit and this voltage causes the motion of Josephson vortices. The state is stationary only at zero interlayer voltage at which there is no current in the load circuit.

One can also arrive at this conclusion in another way. The energy of a single vortex is

$$E_v = \frac{4L_y}{\ell}\sqrt{\frac{2}{\pi}}\sqrt{\rho_s \tilde{t}}. \tag{48}$$

According to equation (48) the pinning may occur due to spatial variation of the tunneling amplitude. In the latter case an approximate solution for the phase can be found by the same way as was done in section 4. The only difference is that the coefficients A_φ, B_φ in equation (34) should be replaced with x-dependent quantities. The solution equation (34) with spatially dependent amplitudes A_φ and B_φ also corresponds to a non-stationary state. It differs from the state described in section 4 in that the phase remains non-stationary in any reference frame, including the frame in which the vortices are at rest. One can show that in this case the dissipative losses are also nonzero and proportional to dissipative coefficients α_φ and $\tilde{\sigma}$.

Thus, nonzero interlayer tunneling results in that the electron–hole pairs in the bilayer cannot transmit electrical energy without dissipation. Nevertheless, since the power of losses is proportional to the square of the tunneling amplitude and this amplitude depends exponentially on the interlayer distance one can expect that it is possible to create bilayer systems with negligibly small dissipation.

References

[1] Eisenstein J P and MacDonald A H 2004 *Nature* **432** 691
[2] Shevchenko S I 1976 *Fiz. Nizk. Temp.* **2** 505
    Shevchenko S I 1976 *Sov. J. Low Temp. Phys.* **2** 251 (Engl. Transl.)
[3] Lozovik Yu E and Yudson V I 1976 *Zh. Eksp. Theor. Fiz.* **71** 738
    Lozovik Yu E and Yudson V I 1976 *Sov. Phys.—JETP* **44** 389 (Engl. Transl.)
[4] MacDonald A H and Rezayi E H 1990 *Phys. Rev.* B **42** 3224
[5] Wen X G and Zee A 1992 *Phys. Rev. Lett.* **69** 1811
[6] Kuramoto Y and Horie C 1978 *Solid State Commun.* **25** 713
[7] Lerner I V and Lozovik Yu E 1981 *Zh. Eksp. Teor. Fiz.* **80** 1488
    Lerner I V and Lozovik Yu E 1981 *Sov. Phys.—JETP* **53** 763 (Engl. Transl.)
[8] Paquet D, Rice T M and Ueda K 1985 *Phys. Rev.* B **32** 5208
[9] Moon K, Mori H, Yang K, Girvin S M, MacDonald A H, Zheng L, Yoshioka D and Zhang S C 1995 *Phys. Rev.* B **51** 5138
[10] Yang K, Moon K, Belkhir L, Mori H, Girvin S M, MacDonald A H, Zheng L and Yoshioka D 1996 *Phys. Rev.* B **54** 11644
[11] Girvin S M and MacDonald A H 1997 Multicomponent quantum hall systems: the sum of their parts and more *Perspectives in Quantum Hall Effects* ed S D Sarma and A Pinczuk (New York: Wiley)
[12] Shevchenko S I 1994 *Phys. Rev. Lett.* **72** 3242
[13] Shevchenko S I 1997 *Phys. Rev.* B **56** 10355
    Shevchenko S I 1998 *Phys. Rev.* B **57** 148
    Bezuglyj A I and Shevchenko S I 2007 *Phys. Rev.* B **75** 075322
[14] Lozovok Yu E and Ruvinsky A M 1997 *Zh. Eksp. Teor. Fiz.* **112** 1791
    Lozovok Yu E and Ruvinsky A M 1997 *JETP* **85** 979 (Engl. Transl.)
[15] Abolfath M, MacDonald A H and Radzihovsky L 2003 *Phys. Rev.* B **68** 155318
[16] Yang K 2001 *Phys. Rev. Lett.* **87** 056802
[17] Joglekar Y N, Balatsky A V and Lilly M P 2005 *Phys. Rev.* B **72** 205313
[18] Kellogg M, Eisenstein J P, Pfeiffer L N and West K W 2004 *Phys. Rev. Lett.* **93** 036801
[19] Tutuc E, Shayegan M and Huse D A 2004 *Phys. Rev. Lett.* **93** 036802
[20] Wiersma R D, Lok J G S, Kraus S, Dietsche W, von Klitzing K, Schuh D, Bichler M, Tranitz H-P and Wegscheider W 2004 *Phys. Rev. Lett.* **93** 266805
[21] Spielman I B, Eisenstein J P, Pfeiffer L N and West K W 2000 *Phys. Rev. Lett.* **84** 5808
    Spielman I B, Kellogg M, Eisenstein J P, Pfeiffer L N and West K W 2004 *Phys. Rev.* B **70** 081303
[22] Spielman I B, Eisenstein J P, Pfeiffer L N and West K W 2001 *Phys. Rev. Lett.* **87** 036803
[23] Kellogg M, Spielman I B, Eisenstein J P, Pfeiffer L N and West K W 2002 *Phys. Rev. Lett.* **88** 126804
    Kellogg M, Eisenstein J P, Pfeiffer L N and West K W 2003 *Phys. Rev. Lett.* **90** 246801
[24] Tiemann L, Lok J G S, Dietsche W, von Klitzing K, Muraki K, Schuh D and Wegscheider W 2008 *Phys. Rev.* B **77** 033306
[25] Tiemann L, Dietsche W, Hauser M and von Klitzing K 2008 *New J. Phys.* **10** 045018
[26] Champagne A R, Finck A D K, Eisenstein J P, Pfeiffer L N and West K W 2008 *Phys. Rev.* B **78** 205310
[27] Butov L V and Filin A I 1998 *Phys. Rev.* B **58** 1980
    Butov L V, Ivanov A L, Imamoglu A, Littlewood P B, Shashkin A A, Dolgopolov V T, Campman K L and Gossard A C 2001 *Phys. Rev. Lett.* **86** 5608
    Butov L V, Lai C W, Ivanov A I, Gossard A C and Chemia D S 2002 *Nature* **417** 47
[28] Larionov A V, Timofeev V B, Hvam J and Soerensen K 2002 *Pisma Zh. Eksp. Teor. Fiz.* **75** 233
    Larionov A V, Timofeev V B, Hvam J and Soerensen K 2002 *JETP Lett.* **75** 200 (Engl. Transl.)
    Dremin A A, Timofeev V B, Larionov A V, Hvam J and Soerensen K 2002 *Pisma Zh. Eksp. Teor. Fiz.* **76** 526
    Dremin A A, Timofeev V B, Larionov A V, Hvam J and Soerensen K 2002 *JETP Lett.* **76** 450 (Engl. Transl.)
[29] Min H, Bistritzer R, Su J J and MacDonald A H 2008 *Phys. Rev.* B **78** 121401
[30] Zhang C H and Joglekar Y N 2008 *Phys. Rev.* B **77** 233405
[31] Berman O L, Lozovik Yu E and Gumbs G 2008 *Phys. Rev.* B **77** 155433

[32] Lozovik Yu E and Sokolik A A 2008 *Pis. Zh. Eksp. Teor. Fiz.*
87 61
Lozovik Yu E and Sokolik A A 2008 *JETP Lett.* **87** 55 (Engl.
Transl.)
[33] Fertig H A and Murthy G 2005 *Phys. Rev. Lett.*
95 156802
[34] Huse D A 2005 *Phys. Rev.* B **72** 064514
[35] Fil D V and Shevchenko S I 2007 *Fiz. Nizk. Temp.* **33** 1023
Fil D V and Shevchenko S I 2007 *Low Temp. Phys.*
33 780 (Engl. Transl.)
[36] Su J J and MacDonald A H 2008 *Nat. Phys.* **4** 799
[37] Simon S B 2005 *Solid State Commun.* **134** 81
[38] Moller G, Simon S H and Rezayi E H 2009 *Phys. Rev.* B
79 125106

[39] Shevchenko S I, Fil D V and Yakovleva A A 2004 *Fiz. Nizk.
Temp.* **30** 431
Shevchenko S I, Fil D V and Yakovleva A A 2004 *Low Temp.
Phys.* **30** 321 (Engl. Transl.)
[40] Kravchenko L Yu and Fil D V 2008 *J. Phys.: Condens. Matter*
20 325235
[41] Iida T and Tsubota M 1999 *Phys. Rev.* B **60** 5802
[42] Putterman S J 1974 *Superfluid Hydrodynamics* (Amsterdam:
North-Holland/American Elsevier)
[43] MacDonald A H, Burkov A A, Joglekar Y N and Rossi E 2004
Physica E **22** 19
[44] Bezuglyj A I and Shevchenko S I 2004 *Fiz. Nizk. Temp.* **30** 282
Bezuglyj A I and Shevchenko S I 2004 *Low Temp. Phys.*
30 208 (Engl. Transl.)